Assignment 12

Coverage: 16.8.

Exercises: 16.8 no 9, 10, 15, 21, 24.

Hand in 16.8 no 10, 15, and Supplementary Problem no 3 by Dec 6.

Supplementary Problems

1. Verify the identity
$$\nabla \times \nabla f = \mathbf{0}$$
for any function f. Use this to show that $x\mathbf{i}+xy\mathbf{j}+xyz\mathbf{k}$ is not conservative in any region.

2. Verify the identity
$$\nabla \cdot \nabla \times \mathbf{F} = 0$$
for any vector field $\mathbf{F}$. Use this fact to show that $x\mathbf{i}+y\mathbf{j}+x^2z\mathbf{k}$ cannot be the curl of some vector field in any region.

3. Let Ω be a region whose boundary is a smooth surface Σ.

   (a) Show that the volume of Ω, V, is given by
$$V = \frac{1}{3}\iint_\Sigma (x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot \mathbf{n}\, d\sigma \ ,$$
   where $\mathbf{n}$ is the outward unit normal at Σ.

   (b) Assume further that Ω is contained in a ball of radius R, deduce from this formula that $V \leq \dfrac{1}{3}R\times S$, where S is the surface area of Σ. Hint: Use the Holder's inequality
$$\left|\iint_\Sigma \mathbf{F}\cdot\mathbf{G}\, d\sigma\right| \leq \sqrt{\iint_\Sigma |\mathbf{F}|^2\, d\sigma}\sqrt{\iint_\Sigma |\mathbf{G}|^2\, d\sigma} \ .$$